

A MOMENT MAP PICTURE OF RELATIVE BALANCED METRICS ON EXTREMAL KÄHLER MANIFOLDS

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ABSTRACT. We give a moment map interpretation of some relatively balanced metrics. As an application, we extend a result of S. K. Donaldson on constant scalar curvature Kähler metrics to the case of extremal metrics. Namely, we show that a given extremal metric is the limit of some specific relatively balanced metrics. As a corollary, we recover uniqueness and splitting results for extremal metrics in the polarized case.

1. INTRODUCTION

In [7], Donaldson gave a general framework to study some specific Fubini-Study metrics called *balanced* metrics on a polarized manifold. It is a finite dimensional counterpart of the moment map interpretation of constant scalar curvature Kähler (cscK, for short) metrics by Fujiki [10] and Donaldson [6]. Donaldson proved that a given cscK metric is the limit of balanced metrics if the automorphism group of the polarized manifold is discrete. In this paper, we extend this framework and its applications to the case of extremal metrics by using some relatively balanced metrics introduced in the authors' previous work [33].

Let (X, ω) be an n -dimensional Kähler manifold. A Kähler metric is called *extremal* in the sense of Calabi [2] if and only if it is a critical point of the functional

$$\omega \mapsto \int_X (S(\omega) - \underline{S})^2 d\mu_\omega$$

defined over the space of Kähler metrics in a given Kähler class, where $S(\omega)$ is the scalar curvature of ω , $d\mu_\omega$ is the volume form $\omega^n/n!$ with respect to ω and \underline{S} is the average of the scalar curvature. These metrics are generalizations of Kähler-Einstein metrics and cscK metrics.

From now on, we consider the case where (X, L) is a polarized manifold, i.e., L is an ample line bundle on X . For an Hermitian metric h on L , let us denote $-i\partial\bar{\partial}\log h$ by ω_h . Then, the metric h induces an inner product $\|\cdot\|_{Hilb_k(h)}$ on $V_k = H^0(X, L^{\otimes k})$ defined by

$$\|s\|_{Hilb_k(h)}^2 = \int_X |s|_{h^k}^2 d\mu_h,$$

where $d\mu_h$ is the volume form with respect to ω_h . Taking an orthonormal basis $\mathbf{s} = \{s_\alpha\}_{\alpha=1}^{N_k}$ of V_k with respect to $Hilb_k(h)$, X can be embedded into \mathbb{CP}^{N_k-1} for k large enough. An Hermitian metric h (or its associated Kähler form ω_h) is called

k^{th} balanced if and only if the pulled-back Fubini-Study metric

$$\omega_{FS_k \circ Hilb_k(h)} = \frac{1}{k} i \partial \bar{\partial} \log \left(\frac{1}{N_k} \sum_{\alpha} |s_{\alpha}|^2 \right) \in 2\pi c_1(L)$$

is equal to ω_h . In [7], Donaldson proved that under the assumption that the group $\text{Aut}(X, L)$ of automorphisms of (X, L) is discrete, if (X, L) admits a cscK metric $\omega_{csc} \in 2\pi c_1(L)$, then there exists a unique k^{th} balanced metric $\omega(k) \in 2\pi c_1(L)$ for each $k \gg 0$ such that $\omega(k)$ converges to ω_{csc} in C^∞ -sense.

Let us drop the discreteness assumption on $\text{Aut}(X, L)$. Let $\text{Aut}_0(X, L)$ be the identity component of $\text{Aut}(X, L)$. Replacing L by a sufficiently large tensor power if necessary, we have a group representation

$$\rho_k : \text{Aut}_0(X, L) \rightarrow \text{SL}(V_k).$$

Fix a maximal torus T in $\text{Aut}_0(X, L)$, its complexification T^c , and denote the image of T^c under ρ_k by T_k^c . As introduced in [33], we call h (or ω_h) k^{th} σ_k -balanced if and only if there exists $\sigma_k \in T_k^c$ such that

$$(1) \quad \omega_{FS_k \circ Hilb_k(h)} = \sigma_k^*(\omega_h).$$

Taking an appropriate orthonormal basis \mathbf{s} of V_k in which σ_k is diagonal, equation (1) is equivalent to the twisted Bergman function

$$(2) \quad \sum_{j=1}^{N_k} e^{-\lambda_j} |s_j|_{h^k}^2$$

being constant on X , where

$$\sigma_k = \exp\left(\frac{1}{2} \text{diag}(\lambda_1, \dots, \lambda_{N_k})\right), \quad \lambda_j \in \mathbb{R}.$$

This is also equivalent to the fact that the embedding of X to \mathbb{CP}^{N_k-1} using \mathbf{s} satisfies

$$\frac{e^{-\frac{1}{2}(\lambda_{\alpha} + \lambda_{\beta})}}{\text{Vol}(X)} \int_X \frac{s_{\alpha} \bar{s}_{\beta}}{\sum_{\gamma} |s_{\gamma}|^2} d\mu_{\omega_{FS_k \circ Hilb_k(h)}} = \delta_{\alpha\beta}.$$

These characterizations tell us that a σ -balanced metric is a specific relative balanced metric as discussed in [21]. Then, the main result of this paper is as follows.

Theorem A. *Let (X, L) be a polarized Kähler manifold with $\omega_{ex} \in 2\pi c_1(L)$ extremal and let T be the identity component of the isometry group of ω_{ex} . Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, (X, L^k) admits a k^{th} σ_k -balanced metric $\omega(k)$ for some $\sigma_k \in T_k^c$. These metrics $\omega(k)$ converge to the initial metric ω_{ex} in C^∞ -sense.*

In order to prove Theorem A, we provide a moment map interpretation of σ -balanced metrics in Section 3. We also provide a characterization of the optimal weights σ_k used in Theorem A in terms of characters on the Lie algebra of T^c .

Remark B. *The choice of the optimal weights σ_k determines a quantization of the extremal vector field, see Section 4.*

Theorem A has the following two applications. First, it slightly simplifies Mabuchi's proof of the uniqueness of extremal metrics on polarized manifolds [22]. Indeed, theorem A allows us to apply directly Kempf-Ness theorem from the theory of moment maps.

Corollary C. *Let (X, L) be a polarized Kähler manifold. An extremal Kähler metric in $2\pi c_1(L)$, if it exists, is unique up to automorphisms of (X, L) .*

Second, it provides a generalization of the splitting theorem of Apostolov-Huang [1].

Corollary D. *Let $(X = X_1 \times X_2, L = L_1 \otimes L_2)$ be a product of polarized Kähler manifold. Assume that X admits an extremal Kähler metric g in the class $2\pi c_1(L)$. Then g is a product metric $g_1 \times g_2$, where g_i is an extremal metric on X_i in the class $2\pi c_1(L_i)$.*

This theorem is proved in [1] with stronger assumptions. In particular, Theorem A was conjectured in [1] to obtain full generality of the above splitting theorem.

We finish this introduction with a brief review on relevant works to Theorem A (see also [1, 17] for comprehensive reviews). The approximation of canonical Kähler metrics by specific Fubini-Study metrics is closely related to the stability of (X, L) in the sense of Geometric Invariant Theory (GIT). In fact, it is well known ([41, 20, 30, 29, 39]) that the existence of a balanced metrics on $(X, L^{\otimes k})$ is equivalent to the Chow stability of the embedding of X to the projective space by sections of $L^{\otimes k}$ if $\text{Aut}(X, L)$ is discrete. The result in [7] implies that if a polarized manifold admits a cscK metric, under the discreteness assumption, then (X, L) is asymptotically Chow stable. This is one of the early evidences for the so-called *Yau-Tian-Donaldson conjecture* which states that the existence of canonical Kähler metrics on a polarized manifold should be equivalent to some stability notion of the manifold in the sense of GIT. Extensions of [7] to the case where $\text{Aut}(X, L)$ is not discrete has been pioneered by Mabuchi [21, 22, 23, 26, 27]. Without the discreteness of $\text{Aut}(X, L)$, even if (X, L) admits cscK metrics, we cannot expect the existence of balanced metrics on $(X, L^{\otimes k})$ for $k \gg 0$. In fact, counter-examples of asymptotic Chow unstable manifolds with cscK metrics are found ([28, 5])¹. This phenomenon comes from the fact that there may exist $v \in \text{Lie}(T_k^c)$ inducing a non-trivial action on the line where the Chow form of $(X, L^{\otimes k})$ lies. This action violates the Chow semistability of $(X, L^{\otimes k})$. To avoid this phenomenon, in [23, 24] (equivalently [12]), the vanishing of some integral invariants is required. However, considering extremal metrics, the above requirement cannot be satisfied, because the action induced by the (non-trivial) extremal vector field violates it. Studying the extension of GIT to the non-discrete case, Mabuchi introduced balanced metrics relative to a given torus in the identity component of the automorphism group of X in [21]. Then, in [25], he proved that the existence of extremal metrics implies the asymptotic existence of relative balanced metrics. A difference between [25] and Theorem A is the choice of the group action on V_k . The group considered in [25] is $\Pi_X SU(N_k^X)$, smaller than (10), considered in the present work. This difference affects the choice of the weight $\{\lambda_j\}$ of relative balanced metrics in (2). In particular, in [25] it is not sure that the weight comes from a torus action. The lack of this information prevents one to apply Székelyhidi's generalization of Kempf-Ness theorem [35]. This is a source of difficulties in applications of [25] to other related problems on extremal metrics. For instance, delicate work is necessary to prove the uniqueness of extremal metrics on polarized manifolds in [22]. Hence, a refinement

¹Theorem A says that on the examples in [28, 5] a cscK metric (i.e., trivial extremal metric) can be approximated by non-trivial σ_k -balanced metrics. In particular, the vector fields induced by σ_k will converges to zero (see Proposition 4.6)

of the results in [25] was expected, e.g. [1]. Very recently, results equivalent to Theorem A are proved by Seyyedali [34] and Mabuchi [27] independently. (Hashimoto gives another quantization of extremal metrics [17].) Let us explain differences between [34, 27] and the proof of Theorem A. While the weight of relative balanced metrics comes from a given extremal metric in [34, 27], we prove that the weight of σ -balanced metrics is determined apriori regardless of the existence of extremal metrics. The latter is a quite natural statement, because the weight of relative balanced metrics approximates the extremal vector field, which exists regardless of the existence of extremal metrics. A motivational observation for our proof is that a σ -balanced metrics is self-similar for some iteration process

$$\omega \mapsto \omega_{FS_k \circ \text{Hilb}_k(\omega)}$$

on \mathcal{H} , called T_k -iteration in [9]. Seeing T_k -iteration as a quantization of the Calabi flow

$$\frac{\partial \varphi_t}{\partial t} = S(\omega_t) - \underline{S}, \quad \omega_t = \omega_0 + i\partial\bar{\partial}\varphi_t,$$

the above observation corresponds to the fact that an extremal metric is a self-similar solution to the Calabi flow. With this point of view, we use an argument analogous to one coming from the theory of Kähler-Ricci solitons [38]. Our strategy is as follows. First, we twist the moment map in [7] by a given σ (Section 3.5). By general theory, it induces a new invariant which is a generalization of the integral invariant considered in [23]. Then we can find the optimal σ so that this new invariant vanishes (Proposition 4.2). Then, the obstruction considered in [23] will vanish in our twisted setting, and we can apply the arguments in [7] and [31] with a slight modification (Section 5).

1.1. Plan of the paper. In Section 2, we collect necessary definitions. In Section 3, we give a moment map interpretation for σ -balanced metrics. In Section 4, we choose the optimal weight σ_k for each $k \gg 0$. In Section 5, we complete the proof of Theorem A, following [7, 31].

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2. SETUP

In this section, we introduce some necessary material and results that will be used throughout the paper.

2.1. Quantization. Let (X, L) be a polarized Kähler manifold of complex dimension n . Let \mathcal{H} be the space of smooth Kähler potentials with respect to a fixed Kähler form $\omega \in 2\pi c_1(L)$:

$$\mathcal{H} = \{\phi \in C^\infty(X) \mid \omega_\phi := \omega + i\partial\bar{\partial}\phi > 0\}.$$

For each k , we can consider \mathcal{H}_k the space of hermitian metrics on $L^{\otimes k}$. To each element $h \in \mathcal{H}_k$ one associates a Kähler metric $-i\partial\bar{\partial}\log h$ on X , identifying the spaces \mathcal{H}_k to \mathcal{H} . Write ω_h to be the curvature of the hermitian metric h on L . Fixing a base metric h_0 in \mathcal{H}_1 such that $\omega = \omega_{h_0}$ the correspondence reads

$$\omega_\phi = \omega_{e^{-\phi}h_0} = \omega + i\partial\bar{\partial}\phi.$$

We denote by \mathcal{B}_k the space of positive definite Hermitian forms on $V_k := H^0(X, L^{\otimes k})$. Let $N_k = \dim V_k$. The spaces \mathcal{B}_k are identified with $GL_{N_k}(\mathbb{C})/U(N_k)$ using the base metric h_0^k . These symmetric spaces come with metrics d_k defined by Riemannian metrics:

$$(H_1, H_2)_h = \text{Tr}(H_1 H^{-1} \cdot H_2 H^{-1}).$$

There are maps :

$$\begin{aligned} \text{Hilb}_k : \mathcal{H} &\rightarrow \mathcal{B}_k \\ FS_k : \mathcal{B}_k &\rightarrow \mathcal{H} \end{aligned}$$

defined by :

$$\forall h \in \mathcal{H}, s \in H^0(X, L^{\otimes k}), \|s\|_{\text{Hilb}_k(h)}^2 = \int_X |s|_{h^k}^2 d\mu_h$$

and

$$\forall H \in \mathcal{B}_k, FS_k(H) = \frac{1}{k} \log \left(\frac{1}{N_k} \sum_{\alpha} |s_{\alpha}|_{h_0^k}^2 \right)$$

where $\mathbf{s} = \{s_{\alpha}\}$ is an orthonormal basis of $H^0(X, L^{\otimes k})$ with respect to H and $d\mu_h = \frac{\omega_h^n}{n!}$ is the volume form. A result of Tian [37] states that any Kähler metric ω_{ϕ} in $2\pi c_1(L)$ can be approximated by projective metrics, namely

$$\lim_{k \rightarrow \infty} FS_k \circ \text{Hilb}_k(\phi) = \phi$$

where the convergence is uniform on $C^2(X, \mathbb{R})$ bounded subsets of \mathcal{H} . The metrics satisfying

$$FS_k \circ \text{Hilb}_k(\phi) = \phi$$

are called balanced metrics. Let $\text{Aut}(X, L)$ be the group of automorphisms of the pair (X, L) . From the work of Donaldson [7], if X admits a cscK metric in the Kähler class $c_1(L)$, and if $\text{Aut}(X, L)$ is discrete, then there are balanced metrics ω_{ϕ_k} for k sufficiently large, with

$$FS_k \circ \text{Hilb}_k(\phi_k) = \phi_k$$

and these metrics converge to the cscK metric on $C^{\infty}(X, \mathbb{R})$ bounded subsets of \mathcal{H} .

In the proof of these results, the density of state function plays a central role. For any $\phi \in \mathcal{H}$ and $k > 0$, let $\{s_{\alpha}\}$ be an orthonormal basis of $H^0(X, L^k)$ with respect to $\text{Hilb}_k(\phi)$. The k^{th} Bergman function of ϕ is defined to be :

$$\rho_k(\phi) = \sum_{\alpha} |s_{\alpha}|_{h^k}^2.$$

It is well known that a metric $\phi \in \text{Hilb}_k(\mathcal{H})$ is balanced if and only if $\rho_k(\phi)$ is constant. A key result in the study of balanced metrics is the following expansion:

Theorem 2.2 ([4],[32],[37],[40]). *The following uniform expansion holds*

$$\rho_k(\phi) = k^n + A_1(\phi)k^{n-1} + A_2(\phi)k^{n-2} + \dots$$

with $A_1(\phi) = \frac{1}{2}S(\phi)$ is half of the scalar curvature of the Kähler metric ω_{ϕ} and for any l and $R \in \mathbb{N}$, there is a constant $C_{l,R}$ such that

$$\|\rho_k(\phi) - \sum_{j \leq R} A_j k^{n-j}\|_{C^l} \leq C_{l,R} k^{n-R}.$$

As a corollary, if $\phi_k = FS_k \circ \text{Hilb}_k(\phi)$, then

$$\phi_k - \phi = \frac{1}{k} \log \rho_k(\phi) \rightarrow 0$$

as $k \rightarrow \infty$. In particular we have the convergence of metrics

$$(3) \quad \omega_{\phi_k} = \omega_\phi + \mathcal{O}(k^{-2}).$$

By integration over X we also deduce

$$\int_X \rho_k(\phi) d\mu_\phi = k^n \int_X d\mu_\phi + k^{n-1} \frac{1}{2} \int_X S(\phi) d\mu_\phi + \mathcal{O}(k^{n-2})$$

where $S(\phi)$ is the scalar curvature of the metric g_ϕ associated to the Kähler form ω_ϕ . Thus

$$(4) \quad N_k = k^n \text{Vol}(X) + \frac{1}{2} \text{Vol}(X) \underline{S} k^{n-1} + \mathcal{O}(k^{n-2}).$$

where

$$\underline{S} = 2n\pi \frac{c_1(-K_X) \cup [\omega]^{n-1}}{[\omega]^n}$$

is the average of the scalar curvature and $\text{Vol}(X)$ is the volume of $(X, c_1(L))$.

2.3. Extremal metrics. In order to find a canonical representative of a Kähler class, Calabi suggested [2] to look for minima of the functional

$$\begin{aligned} Ca : \mathcal{H} &\rightarrow \mathbb{R} \\ \phi &\mapsto \int_X (S(\phi) - \underline{S})^2 d\mu_\phi. \end{aligned}$$

In fact, critical points for this functional are local minima, called extremal metrics. The associated Euler-Lagrange equation is equivalent to the fact that $\text{grad}_{\omega_\phi}(S(\phi))$ is a holomorphic vector field. In particular, cscK metrics are extremal metrics.

By a theorem of Calabi [3], the connected component of identity of the isometry group of an extremal metric is a maximal compact connected subgroup of $\text{Aut}_0(X)$. As all these maximal subgroups are conjugated, the quest for extremal metrics can be done modulo a fixed group action. Note that $\text{Aut}_0(X)$ is isomorphic to the connected component of identity $\text{Aut}_0(X, L)$ of $\text{Aut}(X, L)$. We fix T a maximal torus of $\text{Aut}_0(X, L)$. We define \mathcal{H}^T to be the space of T -invariant potentials with respect to a T -invariant base point ω .

For a fixed metric g , we say that a vector field v is a Hamiltonian vector field if there is a real valued function f such that

$$v = J\nabla_g f$$

or equivalently

$$\omega(v, \cdot) = -df,$$

where J is the almost-complex structure of X .

Remark 2.4. Recall that for any $v \in \text{Lie}(T^c)$ the lift of the action of $\exp(tv)$ on X to L has the following ambiguity (see Remark 2.2 in [14]): for a given Kähler form ω there exists a smooth function $\theta_{v,\omega}$ such that

$$(5) \quad \iota_v \omega = -\bar{\partial} \theta_{v,\omega}$$

uniquely up to the addition of constant. Then, the infinitesimal action of v to L is given by

$$v^\sharp = -2\pi i \theta_{v,\omega} z \frac{\partial}{\partial z} + v^h$$

where z is the fiber coordinate on L and v^h is the horizontal lift with respect to the connection which is equal to ω . Then, the choice of the action on L via $\exp(tv^\sharp)$ has the ambiguity coming from the addition of a constant to $\theta_{v,\omega}$. Hence, we normalize $\theta_{v,\omega}$ so that

$$(6) \quad \int_X \theta_{v,\omega} d\mu_\omega = 0.$$

For any $\phi \in \mathcal{H}^T$, let P_ϕ^T be the space of normalized Killing potentials with respect to g_ϕ whose corresponding Hamiltonian vector fields lie in $\text{Lie}(T)$ and let Π_ϕ^T be the orthogonal projection from $L^2(X, \mathbb{R})$ to P_ϕ^T given by the inner product on functions

$$(7) \quad (f, g) \mapsto \int f g d\mu_\phi.$$

Definition 2.5. [16, Section 4.13] The reduced scalar curvature S^T with respect to T is defined by

$$S^T(\phi) = S(\phi) - \underline{S} - \Pi_\phi^T S(\phi).$$

The extremal vector field v_{ex}^T with respect to T is defined using any $\phi \in \mathcal{H}^T$ by the equation

$$v_{ex}^T = \nabla_g(\Pi_\phi^T S(\phi)).$$

The extremal vector field does not depend on ϕ (see e.g. [16, Proposition 4.13.1]). Once we fix T , we denote v_{ex}^T by v_{ex} , omitting the subscript T . Note that T -invariant metrics satisfying $S^T(\phi) = 0$ are extremal.

3. A MOMENT MAP INTERPRETATION OF σ -BALANCED METRICS

We will consider a generalization of balanced metrics adapted to the relative setting of extremal metrics:

Definition 3.1. Let $\sigma_k(t)$ be a one-parameter subgroup of T_k^c . Let $\phi \in \mathcal{H}$. Then ϕ is a k^{th} σ_k -balanced metric if

$$(8) \quad \omega_{kFS_k \circ \text{Hilb}_k(\phi)} = \sigma_k(1)^* \omega_{k\phi}$$

In this section, we provide a moment map description for σ -balanced metrics. We closely follow the treatment in [16] (see also [1]).

3.2. The relative setting. We extend the quantization tools to the extremal metrics setup. Recall that T is a maximal torus of $\text{Aut}_0(X, L)$. Replacing L by a sufficiently large tensor power if necessary, we can assume that $\text{Aut}_0(X, L)$ acts on L (see e.g. [18]). Then the T -action on X induces a T -action on the space of sections $H^0(X, L^k)$. This action in turn provides a T -action on the space \mathcal{B}_k of positive definite hermitian forms on $H^0(X, L^k)$ and we define \mathcal{B}_k^T to be the subspace of

T -invariant elements. The spaces \mathcal{B}_k^T are totally geodesic in \mathcal{B}_k for the distances d_k . We see from their definitions that we have the induced maps :

$$(9) \quad \begin{aligned} \text{Hilb}_k &: \mathcal{H}^T \rightarrow \mathcal{B}_k^T \\ FS_k &: \mathcal{B}_k^T \rightarrow \mathcal{H}^T. \end{aligned}$$

Since $\text{Aut}_0(X, L)$ acts on L , we have a group representation

$$\rho_k : \text{Aut}_0(X, L) \rightarrow \text{SL}(V_k).$$

The action of the complexified torus $T_k^c := \rho_k(T^c)$ on V_k induces a weight decomposition

$$V_k = \bigoplus_{\chi \in w_k(T)} V_k(\chi)$$

where $\rho_k(T^c)$ acts on $V_k(\chi)$ with weight χ , and $w_k(T)$ is the space of weights for this action. Let N_k^χ be the dimension of $V_k(\chi)$.

We will be interested in the space \mathcal{B}_k^T of T -invariant metrics on V_k . We consider the corresponding space of basis of V_k

$$\mathcal{B}^T(V_k) := \left\{ (s_i^\chi)_{\chi \in w_k(T); i=1..N_k^\chi} \in (V_k)^{N_k} \mid \det(s_i^\chi) \neq 0 \text{ and } \forall (\chi, i), s_i^\chi \in V_k(\chi) \right\}$$

For each k , we define the subgroup

$$\text{Aut}_k^T \subset \text{SL}(V_k)$$

to be the centralizer of T_k^c in $\rho_k(\text{Aut}_0(X))$ and the space $Z^T(V_k)$ to be the quotient

$$Z^T(V_k) = \mathcal{B}^T(V_k) / (\mathbb{C}^* \times \text{Aut}_k^T),$$

where \mathbb{C}^* acts by scalar multiplication. Then consider the group

$$G_k^c = S(\Pi_\chi GL_{N_k^\chi}(\mathbb{C}))$$

which is the complexification of

$$(10) \quad G_k := S(\Pi_\chi U(N_k^\chi)).$$

At each element $\mathbf{s} \in \mathcal{B}^T(V_k)$, we define an isomorphism

$$\Phi_{\mathbf{s}} : \mathbb{P}(V_k^*) \rightarrow \mathbb{CP}^{N_k-1}.$$

There is a natural right action of G_k^c on $\mathcal{B}^T(V_k)$ that commutes with the left action of $\mathbb{C}^* \times \text{Aut}_k^T$ on $\mathcal{B}^T(V_k)$. Then the actions of these groups descend to actions on the quotient $Z^T(V_k)$. We will see in the next section that the space $Z^T(V_k)$ carries a symplectic structure (in fact Kähler structure) such that the σ_k -balanced condition appears as the vanishing of the moment map with respect to the action of G_k .

3.3. A Kähler structure on $Z^T(V_k)$ for weighted considerations. In this section, we will abbreviate the subscript k if it does not lead to confusion. Fix an element $\sigma \in T^c \subset \text{SL}(V)$. As a space of basis for a complex vector space, $\mathcal{B}^T(V)$ carries a natural integrable almost-complex structure $J_{\mathcal{B}}$ that descends to an integrable almost-complex structure J_Z on the quotient $Z^T(V)$. Then we build a symplectic form as follows. First of all, to each $\mathbf{s} \in \mathcal{B}^T(V)$ we can associate a unique element $H(\mathbf{s}) \in \mathcal{B}^T$ so that \mathbf{s} is an orthonormal basis of $H(\mathbf{s})$. Note also that there is a map:

$$\begin{aligned} \phi : \mathcal{B}^T(V) &\rightarrow \mathcal{H}^T \\ \mathbf{s} &\mapsto FS(H(\mathbf{s})), \end{aligned}$$

where

$$FS(H(\mathbf{s})) = \frac{1}{k} \log \left(\frac{1}{N_k} \sum_{\alpha} |s_{\alpha}|_{h_0^k}^2 \right)$$

for $\mathbf{s} = \{s_{\alpha}\}$ and h_0 is a fixed Hermitian metric on L . We will sometimes write $\phi_{\mathbf{s}}$ for $\phi(\mathbf{s})$. If $\iota : X \hookrightarrow \mathbb{P}(V^*)$ denotes the Kodaira embedding, then $\omega_{\phi_{\mathbf{s}}} = (\Phi_{\mathbf{s}} \circ \iota)^* \omega_{FS}$.

We introduce a twisted Kähler form on $\mathcal{B}^T(V)$ by σ as follows. Take $v \in \text{Lie}(T^c)$ so that $\exp(v) = \sigma$. For a given metric ω_{ϕ} , define the function $\psi_{\sigma, \phi}$ by

$$(11) \quad \sigma^* \omega_{\phi} = \omega_{\phi} + i \partial \bar{\partial} \psi_{\sigma, \phi}$$

with the normalization

$$(12) \quad \int_X \exp(\psi_{\sigma, \phi}) d\mu_{\phi} = \frac{N_k}{k^n}.$$

Then we consider a modified Aubin functional introduced in [33] defined up to a constant by its differential:

$$dI^{\sigma}(\phi)(\delta\phi) = \int_X \delta\phi(1 + \Delta_{\phi}) e^{\psi_{\sigma, \phi}} d\mu_{\phi}$$

where $\Delta_{\phi} = -g_{\phi}^{i\bar{j}} \frac{\partial}{\partial z_i} \frac{\partial}{\partial \bar{z}_j}$ is the complex Laplacian of g_{ϕ} . This one-form integrates along paths in the space \mathcal{H}^T of T -invariant Kähler potentials to a functional $I^{\sigma}(\phi)$ on \mathcal{H}^T , which is independent on the path used from 0 to ϕ (see [33]). Define the form on $\mathcal{B}^T(V)$ by

$$\omega_{\mathcal{B}}(\mathbf{s}) := dd^c(I^{\sigma} \circ \phi(\mathbf{s}))$$

where d^c is defined with respect to $J_{\mathcal{B}}$. Then we prove the following:

Proposition 3.4. *The 2-form $\omega_{\mathcal{B}}$ descends to a G -invariant symplectic form ω_Z on $Z^T(V)$ such that $(Z^T(V), J_Z, \omega_Z)$ is Kähler.*

Proof. Let denote by $\theta_{\mathcal{B}}$ the 1-form $d^c(I^{\sigma}(\phi))$. We show that $\theta_{\mathcal{B}}$ is invariant under the actions of G and Aut^T . Then $\theta_{\mathcal{B}}$ and $\omega_{\mathcal{B}} = d\theta_{\mathcal{B}}$ descend to G -invariant forms on $Z^T(V)$. By definition, for any $\mathbf{s} \in \mathcal{B}^T(V)$ and any $A \in T_{\mathbf{s}}\mathcal{B}^T(V)$,

$$\theta_{\mathcal{B}}(\mathbf{s})(A) = d^c(I^{\sigma} \circ \phi(\mathbf{s}))(A) = -d(I^{\sigma} \circ \phi(\mathbf{s}))(iA) = -dI^{\sigma}(\phi_{\mathbf{s}})(D_{\mathbf{s}}\phi(iA))$$

and thus

$$\theta_{\mathcal{B}}(\mathbf{s})(A) = \int_X D_{\mathbf{s}}\phi(iA)(1 + \Delta_{\phi_{\mathbf{s}}}) e^{\psi_{\sigma, \phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}}.$$

Note that at each point $\mathbf{s} \in \mathcal{B}^T(V)$, the isomorphism $\Phi_{\mathbf{s}}$ gives an isomorphism

$$T_{\mathbf{s}}\mathcal{B}^T(V) \simeq \bigoplus_{\chi} \mathfrak{gl}_{N_{\chi}}(\mathbb{C}).$$

We then denote by \hat{A} the vector field on \mathbb{CP}^N induced by $A = (A_{ij}) \in T_{\mathbf{s}}\mathcal{B}^T(V)$. Then, a direct computation shows (e.g. [16, lemma 8.4.3])

$$D_{\mathbf{s}}\phi(iA) = -\frac{\sum_{ij} A_{ij}^{-}(s_i, s_j)_{h_0}}{\sum_k |s_k|_{h_0}^2}$$

where A^{-} is the anti-hermitian part of A . In other words, if $\mu^{\hat{A}^{-}}$ denotes the momentum of \hat{A}^{-} on \mathbb{CP}^N ,

$$D_{\mathbf{s}}\phi(iA) = -\mu^{\hat{A}^{-}} \circ \Phi_{\mathbf{s}} \circ \iota.$$

To simplify notations, we let $\mu_{\mathbf{s}}^{\hat{A}^-} = \mu^{\hat{A}^-} \circ \Phi_{\mathbf{s}} \circ \iota$. We thus found

$$(13) \quad \theta_{\mathcal{B}}(\mathbf{s})(A) = - \int_X \mu_{\mathbf{s}}^{\hat{A}^-} (1 + \Delta_{\phi_{\mathbf{s}}}) e^{\psi_{\sigma, \phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}}.$$

Now from the definition of the action of G on $\mathcal{B}^T(V)$,

$$\forall g \in G, \omega_{\phi(\mathbf{s} \cdot g)} = \omega_{\phi(\mathbf{s})}.$$

Thus we see that $\theta_{\mathcal{B}}$ is G -invariant. Then ([16] Proposition 8.3.2)

$$\forall \gamma \in \text{Aut}^T, \omega_{\phi(\gamma \cdot \mathbf{s})} = \gamma^* \omega_{\phi(\mathbf{s})},$$

and a change of variables in (13) shows that $\theta_{\mathcal{B}}$ is Aut^T -invariant.

It remains to show that $g_{\mathcal{B}} := \omega_{\mathcal{B}}(\cdot, J_{\mathcal{B}} \cdot)$ is positive and vanishes exactly on the distribution given by the leaves of the $\mathbb{C}^* \times \text{Aut}^T$ -orbits. Let $\mathbf{s} \in \mathcal{B}^T(V)$ and $A \in T_{\mathbf{s}}\mathcal{B}^T(V)$. We have

$$g_{\mathcal{B}}(A, A) = \omega_{\mathcal{B}}(A, J_{\mathcal{B}} A) = dd^c(I^\sigma \circ \phi)(A, J_{\mathcal{B}} A).$$

Let $\mathbf{s}(t) = \mathbf{s} \cdot \exp(tA)$ and $\mathbf{s}(t)^c = \mathbf{s} \cdot \exp(tJ_{\mathcal{B}} A)$. Then, as $J_{\mathcal{B}}$ is integrable,

$$(14) \quad g_{\mathcal{B}}(A, A) = \frac{d^2}{dt^2} \Big|_{t=0} (I^\sigma \circ \phi)(\mathbf{s}(t)) + \frac{d^2}{dt^2} \Big|_{t=0} (I^\sigma \circ \phi)(\mathbf{s}(t)^c).$$

If $A \in T_{\mathbf{s}}\mathcal{B}^T(V) \simeq \bigoplus_{\chi} \mathfrak{gl}_{N_{\chi}}(\mathbb{C})$ is diagonal, that is to say, corresponds to the \mathbb{C}^* action on $\mathcal{B}^T(V)$, then we easily compute $g_{\mathcal{B}}(A, A) = 0$. Now if we assume A to be trace-free, by [33, Lemma 3.3.1] applied to

$$\phi(\mathbf{s}(t)) = \log\left(\sum_{\alpha} |s_{\alpha} \cdot \exp(tA)|_{h_0}^2\right)$$

and

$$\phi(\mathbf{s}(t)^c) = \log\left(\sum_{\alpha} |s_{\alpha} \cdot \exp(tJ_{\mathcal{B}} A)|_{h_0}^2\right),$$

we deduce that

$$g_{\mathcal{B}}(A, A) \geq 0$$

with equality if and only if $t \mapsto \exp(tA)$ and $t \mapsto \exp(tJ_{\mathcal{B}} A)$ (or more precisely the subgroups of $SL(V)$ determined by $\mathbf{s}(t)$ and $\mathbf{s}(t)^c$) are in Aut^T . This concludes the proof. \square

3.5. The moment map for weighted considerations. We define

$$\begin{aligned} \mu^{\sigma} : \mathcal{B}^T(V) &\rightarrow \text{Lie}(G) \\ \mathbf{s} &\mapsto i\text{Hilb}(\phi(\mathbf{s}))_0(\sigma \cdot s_j, \sigma \cdot s_k) \end{aligned}$$

where the subscript 0 means the trace-free part of the matrix. The σ -balanced condition correspond to the existence of a basis $\mathbf{s} \in (\mu^{\sigma})^{-1}(0)$:

Proposition 3.6. *Let $\mathbf{s} \in \mathcal{B}^T(V)$. Then $\omega_{\phi(\mathbf{s})}$ is σ -balanced if and only if $\mu^{\sigma}(\mathbf{s}) = 0$.*

Proof. Let $\mathbf{s} = (s_j) \in \mathcal{B}^T(V)$. By definition,

$$\begin{aligned} (15) \quad [\mu^{\sigma}(\mathbf{s})]_{jk} &= i\text{Hilb}(\phi(\mathbf{s}))_0(\sigma^* s_j, \sigma^* s_k) \\ &= i \left(\int_X (\sigma^* s_j, \sigma^* s_k)_{h_{\phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}} - c\delta_{jk} \right) \end{aligned}$$

for some constant $c \neq 0$. Then $\mu^{\sigma}(\mathbf{s}) = 0$ if and only if $c^{-1/2}\sigma^*\mathbf{s}$ is an orthonormal basis for $\text{Hilb}(\phi(\mathbf{s}))$. This is the same as $\omega_{\phi_{\mathbf{s}}}$ being σ -balanced. \square

Lastly, we show that μ^σ is indeed a moment map for our setup:

Proposition 3.7. *The map μ^σ descends to an equivariant moment map for the G -action on $Z^T(V)$ with respect to ω_Z .*

Proof. To see μ^σ as a moment map, we identify $\text{Lie}(G)$ with its dual using the standard hermitian product on $\mathfrak{su}_N(\mathbb{C})$. First, note that μ^σ takes value in $\text{Lie}(G)$ because $\phi(\mathbf{s})$ is T -invariant, and σ commutes with T so that $\text{Hilb}(\sigma^*\phi(\cdot))$ maps $\mathcal{B}^T(V)$ to $\mathcal{B}^T \simeq \Pi_\chi GL_{N^\times}(\mathbb{C})/U(N^\times)$. Note that $\text{Lie}(G) = (\bigoplus_\chi \mathfrak{u}(N^\times))_0$ and the trace-free part of $\Pi_\chi GL_{N^\times}(\mathbb{C})/U(N^\times)$ is $i\text{Lie}(G)$.

Let $\mathbf{s} = (s_j) \in \mathcal{B}^T(V)$. Any element $a \in \text{Lie}(G)$ defines a vector field \hat{a} on $\mathcal{B}^T(V)$ via $\Phi_{\mathbf{s}}$. We now want to prove

$$(16) \quad \forall a \in \text{Lie}(G), \quad \langle \mu^\sigma(\mathbf{s}), a \rangle = \theta_{\mathcal{B}}(\mathbf{s})(\hat{a})$$

The left hand term in (16) is

$$\begin{aligned} \langle \mu^\sigma(\mathbf{s}), a \rangle &= \langle i\text{Hilb}(\sigma^*\phi(\mathbf{s}))_0, a \rangle \\ &= i \int_X \langle [(\sigma^*s_k, \sigma^*s_j)_{h_{\phi_{\mathbf{s}}}}]_{kj}, a \rangle d\mu_{\phi_{\mathbf{s}}} \\ (17) \quad &= - \int_X \frac{\sum_{kj} (\sigma^*s_k, \sigma^*s_j)_{h_0} \overline{ia_{jk}}}{\sum_k |s_k|_{h_0}^2} d\mu_{\phi_{\mathbf{s}}}. \end{aligned}$$

On the other hand, from (13),

$$(18) \quad \theta_{\mathcal{B}}(\mathbf{s})(\hat{a}) = - \int_X \mu_{\mathbf{s}}^a (1 + \Delta_{\phi_{\mathbf{s}}}) e^{\psi_{\sigma, \phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}}.$$

Then, from the proof of Lemma 3.3.1 in [33], applying [33, Equation (15)] to the path of metrics $\phi_t = \phi(\mathbf{s} \cdot e^{tiA})$ in \mathcal{H}^T , we have:

$$(19) \quad \int_X \mu_{\mathbf{s}}^a (1 + \Delta_{\phi_{\mathbf{s}}}) e^{\psi_{\sigma, \phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}} = \int_X \frac{\sum_{jk} (\sigma^*s_k, \sigma^*s_j)_{h_0} \overline{ia_{jk}}}{\sum_k |s_k|_{h_0}^2} d\mu_{\phi_{\mathbf{s}}}.$$

By (17), (18) and (19), we deduce (16). From Proposition 3.4, we see that $\theta_{\mathcal{B}}(\hat{a})$ is the momentum for \hat{a} on $Z^T(V)$. As $\omega_{\mathcal{B}} = d\theta_{\mathcal{B}}$, the only thing that remains to show is that μ^σ is equivariant with respect to the G -action. Let $g \in G$. By (15),

$$[\mu^\sigma(\mathbf{s} \cdot g)]_{jk} = i\text{Hilb}(\phi(\mathbf{s} \cdot g))_0(\sigma^*(s_j \cdot g), \sigma^*(s_k \cdot g)),$$

and as $\omega_{\phi(\mathbf{s} \cdot g)} = \omega_{\phi(\mathbf{s})}$,

$$\begin{aligned} [\mu^\sigma(\mathbf{s} \cdot g)]_{jk} &= i\text{Hilb}(\phi(\mathbf{s}))_0(\sigma^*(s_j \cdot g), \sigma^*(s_k \cdot g)) \\ &= i\text{Hilb}(\phi(\mathbf{s}))_0((\sigma^*s_j) \cdot g, (\sigma^*s_k) \cdot g). \end{aligned}$$

The last equality comes from the fact that the right and left actions of G and Aut^T commute. Then

$$\mu^\sigma(\mathbf{s} \cdot g) = \text{Ad}_{g^{-1}} [i\text{Hilb}(\phi(\mathbf{s}))_0(\sigma^*s_j, \sigma^*s_k)] = \text{Ad}_{g^{-1}}(\mu^\sigma(\mathbf{s})).$$

Thus μ^σ is Ad -equivariant, and defines a moment map for the G -action on $Z^T(V)$. \square

4. OPTIMAL CHOICE OF THE WEIGHT σ

Through this section, we abbreviate the subscript k if it does not lead to confusion. With a moment map interpretation for σ -balanced metrics at hand, we would like to show that for k large enough, there is a σ -balanced metric on X , i.e., the zero of the moment map μ^σ . To find such a point, we consider the gradient flow of $I^\sigma \circ \phi$. General theory of moment maps reduces the problem to the estimate of lower bound of the first derivative of μ^σ . If T is not trivial, we cannot hope to get the desired estimate for general $\sigma \in T^c$. Hence, we need to choose σ carefully to avoid this obstruction. Our argument is inspired by [38] from the viewpoint that σ -balanced metrics are self-similar for the discrete dynamical system

$$\omega \mapsto \omega_{FS \circ \text{Hilb}(\omega)}.$$

4.1. Optimal weight σ . First, we introduce an invariant from the derivative of μ^σ , which determines the optimal weight σ . Fix any $\mathbf{s} \in \mathcal{B}^T(V)$, consider the corresponding Kähler metric $\omega_{\phi_{\mathbf{s}}}$. Take $a = (a_{ij}) \in \text{Lie}(T^c)$ so that $\{\exp(ita)\} \subset SL(V)$. Define the function $\theta_{a,\mathbf{s}}$ by

$$\theta_{a,\mathbf{s}} := \frac{\sum_{jk} (s_j, s_k)_{h_0} \overline{ia_{jk}}}{\sum_j |s_j|_{h_0}^2}.$$

where $\mathbf{s} = \{s_j\}$. For $\sigma \in T^c$, similarly to the modified Futaki invariant, we define the map $\mathcal{F}^\sigma : \text{Lie}(T^c) \rightarrow \mathbb{C}$ by

$$\mathcal{F}^\sigma(a) = - \int_X \theta_{a,\mathbf{s}} (1 + \Delta_{\omega_{\phi_{\mathbf{s}}}}) e^{\psi_{\sigma,\phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}}.$$

Proposition 4.2. *The map \mathcal{F}^σ satisfies:*

(1) *If \mathbf{s} is σ -balanced, then \mathcal{F}^σ vanishes. More precisely:*

$$\frac{d}{dt} \Big|_{t=0} I^\sigma \circ \phi(\mathbf{s} \cdot e^{ita}) = \langle \mu^\sigma(\mathbf{s}), a \rangle = \mathcal{F}^\sigma(a).$$

(2) *\mathcal{F}^σ is independent of the choice of basis \mathbf{s} .*

(3) *There exists a unique $\sigma \in T^c$, independent of the choice of \mathbf{s} , such that \mathcal{F}^σ vanishes.*

Definition 4.3. The weight $\sigma_k \in T^c$ such that $\mathcal{F}^{\sigma_k} = 0$ will be called the k^{th} *optimal weight* (or optimal weight).

Proof. The statement (1) follows directly by definition. For an element $a \in \text{Lie}(T^c)$, define the function $\tilde{\theta}_{a,\mathbf{s}}$

$$(20) \quad \iota_{\hat{a}} \omega_{\phi_{\mathbf{s}}} = i \bar{\partial} \tilde{\theta}_{a,\mathbf{s}}, \quad \int_X \tilde{\theta}_{a,\mathbf{s}} d\mu_{\phi_{\mathbf{s}}} = 0.$$

It is known that $\theta_{a,\phi_{\mathbf{s}}} - \tilde{\theta}_{a,\phi_{\mathbf{s}}}$ is equal to a constant which is independent of the choice of \mathbf{s} (cf. the proof of Lemma 3.4 [12]). More precisely, it is equal to the integral invariant of Futaki-Morita type [15]

$$\int_X c_1^{n+1}(A(a_*) + F_A)$$

where A and F_A are a connection form of type $(1,0)$ and its curvature form of the associated \mathbb{C}^* -bundle $\mathbb{P}_{\mathbb{C}^*}$ of L with the left action induced by a respectively, and

a_* is the associated vector field on $\mathbb{P}_{\mathbb{C}^*}$ induced by a . Then, it is sufficient to show that

$$\tilde{\mathcal{F}}^\sigma(a) = - \int_X \tilde{\theta}_{a,\mathbf{s}} (1 + \Delta_{\omega_{\phi_{\mathbf{s}}}}) e^{\psi_{\sigma,\phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}}$$

is independent of the choice of \mathbf{s} . Take another basis \mathbf{s}' and connect it with \mathbf{s} by a geodesic $\{\exp(it\xi) \mid t \in [0, 1]\}$. At this moment, we omit subscriptions if it does not lead to confusion. By direct calculation,

$$\begin{aligned} -\frac{d}{dt}\tilde{\mathcal{F}}^\sigma(a) &= \int_X ((1 + \Delta)\dot{\tilde{\theta}}) e^\psi d\mu + \int_X (\nabla \bar{\nabla} \dot{\phi}, \nabla \bar{\nabla} \tilde{\theta}) e^\psi d\mu \\ &\quad + \int_X ((1 + \Delta)\dot{\tilde{\theta}}) (\nabla \psi, \nabla \dot{\phi}) e^\psi d\mu - \int_X ((1 + \Delta)\dot{\tilde{\theta}}) e^\psi \Delta \dot{\phi} d\mu \\ &= \int_X ((1 + \Delta)\dot{\tilde{\theta}}) e^\psi d\mu - \int_X (\nabla \dot{\phi}, \nabla \tilde{\theta}) e^\psi d\mu - \int_X (\nabla \bar{\nabla} \tilde{\theta}, \nabla \dot{\phi} \bar{\nabla} \psi) e^\psi d\mu \\ (21) \quad &= \int_X (1 + \Delta)(\dot{\tilde{\theta}} - (\nabla \dot{\phi}, \nabla \tilde{\theta})) e^\psi d\mu. \end{aligned}$$

Above, we used the equality (cf. Lemma 4.2 [33])

$$\dot{\psi} = (\nabla \psi, \nabla \dot{\phi}).$$

The equality (20) implies that $\dot{\tilde{\theta}} - (\nabla \dot{\phi}, \nabla \tilde{\theta})$ is constant on X . The second equality in (20) that the constant is zero, in fact

$$\int_X (\dot{\tilde{\theta}} - (\nabla \dot{\phi}, \nabla \tilde{\theta})) d\mu = \frac{d}{dt} \int_X \tilde{\theta} d\mu = 0.$$

Hence, $\frac{d}{dt}\tilde{\mathcal{F}}^\sigma(a)$ is independent of t . The proof of (2) is completed. We will show (3). Assume that a is diagonal by a change of \mathbf{s} . We denote $\tau(t) = \exp(ita)$. By definition,

$$\begin{aligned} \mathcal{F}^\sigma(a) &= - \int_X ((1 + \Delta_{\omega_{\phi_{\mathbf{s}}}}) \theta_{a,\mathbf{s}}) e^{\psi_{\sigma,\phi_{\mathbf{s}}}} d\mu_{\phi_{\mathbf{s}}} \\ &= \frac{d}{dt} \Big|_{t=0} \int_X \frac{\sum_i |\sigma^* s_i|^2}{\sum_j |s_j \cdot \tau(t)|^2} d\mu_{\phi_{\mathbf{s} \cdot \tau(t)}} \\ &= \frac{d}{dt} \Big|_{t=0} \int_X \frac{\sum_i |s_i \cdot \sigma \cdot \tau(-t)|^2}{\sum_j |s_j|^2} d\mu_{\phi_{\mathbf{s}}}, \end{aligned}$$

because $\tau(t)$ commutes with σ . For a given \mathbf{s} , we define the functional $\mathcal{G} : T^c \rightarrow \mathbb{R}$ by

$$\mathcal{G}(\sigma) := \int_X \frac{\sum_i |s_i \cdot \sigma|^2}{\sum_j |s_j|^2} d\mu_{\mathbf{s}}.$$

By direct calculation, we find that \mathcal{G} is independent of \mathbf{s} . Obviously, \mathcal{G} is strongly convex and proper on T^c . Hence, \mathcal{G} has a unique critical point $\sigma_0 \in T^c$ independent of the choice of \mathbf{s} . Considering the first variation of \mathcal{G} , we find that \mathcal{F}^{σ_0} vanishes on $\text{Lie}(T^c)$. The proof is completed. \square

Remark 4.4. If $\sigma = \text{id}$, then the vanishing of \mathcal{F}^{id} is equivalent to the condition in Theorem A [23] called stability of isotropy actions for (X, L) . It is also equivalent to the vanishing of higher Futaki invariants [12].

4.5. Convergence of the optimal weights σ_k . Next, we will show that the optimal weight σ_k for each k approximates the extremal vector field v_{ex} .

Proposition 4.6. *Let $\sigma_k \in T_k^c$ be the optimal weight as above. Take $v_k \in \text{Lie}(T^c)$ so that $\exp(v_k) = \sigma_k$. Then, we have*

$$\lim_{k \rightarrow \infty} kv_k = v_{ex}$$

where v_{ex} is the extremal vector field of (X, L) with respect to the torus T .

Proof. First we construct an approximate weight for each k .

Lemma 4.7. *There exist a constant $c \in \mathbb{R}$ and vector fields $\nu_j \in \text{Lie}(T^c)$ ($j \geq 2$) such that for any $q \geq 2$ there exists a constant C_q with the following property: let*

$$(22) \quad v_q(k) := cv_{ex}k^{-1} + \sum_{j=2}^q \nu_j k^{-j},$$

then we have

$$(23) \quad |\mathcal{F}^{\tilde{\sigma}_k}(u)| < C_q k^{-q-1} \max_i |b_i| \quad \text{for } k \gg 0$$

where $\tilde{\sigma}_k := \exp(v_q(k)) \in T_k^c$ and $u \in \text{Lie}(T^c)$ is written by

$$u = \text{diag} \frac{1}{2}(b_1, \dots, b_{N_k}) \in \text{Lie}(T_k^c)$$

with respect to $\text{Hilb}_k(\omega)$.

Proof. We denote by ω_k the Kähler form $\omega_{FS_k \circ \text{Hilb}_k(\omega)}$ through the proof. For a given q and a vector field in the form as (22)

$$v_q(k) = \sum_{j=1}^q \nu_j k^{-j}$$

let

$$\tilde{\sigma}_k(t) := \exp(tv_q(k))$$

and $\tilde{\sigma}_k := \exp(1 \cdot v_q(k))$. Take any $u \in \text{Lie}(T^c)$ and let $\tau_k(t) := \exp(tu) \in T_k^c$. From (12), (20) and that $\theta_{u, \omega_k} - \tilde{\theta}_{u, \omega_k}$ is constant, we find that

$$(24) \quad \begin{aligned} \mathcal{F}^{\tilde{\sigma}_k}(u) - \tilde{\mathcal{F}}^{\tilde{\sigma}_k}(u) &= - \int_X (\theta_{u, \omega_k} - \tilde{\theta}_{u, \omega_k}) e^{\psi_{\tilde{\sigma}_k, \omega_k}} d\mu_{\omega_k} \\ &= - \frac{N_k}{k^n} \frac{1}{\text{vol}(X)} \int_X (\theta_{u, \omega_k} - \tilde{\theta}_{u, \omega_k}) d\mu_{\omega_k} \\ &= - \frac{N_k}{k^n} \frac{1}{\text{vol}(X)} \int_X \theta_{u, \omega_k} d\mu_{\omega_k} \end{aligned}$$

for each k . We now give some expansions for $\tilde{\mathcal{F}}^{\tilde{\sigma}_k}(u)$ and $\frac{1}{\text{vol}(X)} \int_X \theta_{u, \omega_k} d\mu_{\omega_k}$. Note that for any 1-parameter subgroup $\{\sigma(t) = \exp(t\nu)\} \subset T_k^c$, we have an expansion

$$(25) \quad \sigma(t)^* \omega_k = \omega_k + i\partial\bar{\partial} \left(\sum_{j=1}^q \tilde{\theta}_{\nu, \omega_k}^j t^j \right) + \mathcal{O}(t^{q+1})$$

with

$$\int_X \tilde{\theta}_{\nu, \omega_k}^j d\mu_{\omega_k} = 0, \quad j = 1 \dots q.$$

As the vector fields ν_j commute, so do the automorphisms $\exp(\nu_j k^{-j})$. Thus, by definition of $v_q(k)$, and using (25),

$$\begin{aligned} i\partial\bar{\partial}\psi_{\tilde{\sigma}_k, \omega_k} &= \left(\frac{1}{k}\right) \frac{(\tilde{\sigma}_k(1/k))^* \omega_k - \omega_k}{k^{-1}} \\ (26) \quad &= i\partial\bar{\partial}\left(\sum_{j=1}^q (\tilde{\theta}_{\nu_j, \omega_k} + \theta'_j) k^{-j}\right) + \mathcal{O}(k^{-q-1}), \end{aligned}$$

where

$$\iota_{\nu_j} \omega_k = i\partial\bar{\partial}\tilde{\theta}_{\nu_j, \omega_k}, \quad \int_X \tilde{\theta}_{\nu_j, \omega_k} d\mu_{\omega_k} = 0$$

and the functions θ'_j , $j = 1 \dots q$, only depend on ω_k and the vector fields $(\nu_1, \dots, \nu_{j-1})$. Then, we have

$$(27) \quad \tilde{\mathcal{F}}^{\tilde{\sigma}_k}(u) = - \int_X \tilde{\theta}_{u, \omega_k} (1 + \Delta_{\omega_k}) \left(1 + \sum_{j=1}^q (\tilde{\theta}_{\nu_j, \omega_k} + P_j(\tilde{\theta}_{\nu_i, \omega_k}, \theta'_i)) k^{-j}\right) d\mu_{\omega_k} + \mathcal{O}(k^{-q-1})$$

with $P_j(\tilde{\theta}_{\nu_i, \omega_k}, \theta'_i)$ a polynomial in $(\tilde{\theta}_{\nu_1, \omega_k}, \dots, \tilde{\theta}_{\nu_{j-1}, \omega_k}, \theta'_1, \dots, \theta'_{j-1})$. Note that the coefficients of k^{-j} ($j \geq 1$) in (27) are independent of the choice of ω . This follows from the calculation (21). In fact, denoting the expansion

$$\exp(\Psi_{\sigma_k, \omega_k}) = 1 + \sum_{j \geq 1} \tilde{\Theta}_j k^{-j},$$

then the calculation (21) tells us

$$\begin{aligned} \frac{d}{ds} \int_X \tilde{\theta}_{u, \omega_k} (1 + \Delta_{\omega_k}) \tilde{\Theta}_j d\mu_{\omega_k} &= \int_X (1 + \Delta_k) \left(\frac{\partial \tilde{\theta}_{u, \omega_k}}{\partial s} - (\nabla \frac{\partial \phi}{\partial s}, \nabla \tilde{\theta}_{u, \omega_k}) \right) \tilde{\Theta}_j d\mu_{\omega_k} \\ &= 0 \end{aligned}$$

where s parametrizes a perturbation of the Kähler form. On the other hand, according to Proposition 2.2.2 in [8], we find that

$$(28) \quad \frac{1}{\text{vol}(X)} \int_X \theta_{u, \omega_k} d\mu_{\omega_k} = F_1(u) k^{-1} + F_2(u) k^{-2} + \dots + F_q(u) k^{-q} + \mathcal{O}(k^{-q-1})$$

where F_j is a Lie algebra homomorphism on $\text{Lie}(T^c)$ to \mathbb{R} . In particular, F_1 is equal to the original Futaki invariant [11]

$$\text{Fut}(u) = \int_X \tilde{\theta}_{u, \omega} S(\omega) d\mu_{\omega}$$

up to multiplication by a constant. Let us recall how to get (28). From the equivariant Riemann-Roch theorem, the weight w_k of $\tau_k(t)$ on V_k is given by

$$w_k = \left(\int_X \theta_{u, \omega_k} d\mu_{\omega_k} \right) k^{n+1} + c' \left(\int_X \theta_{u, \omega_k} S(\omega_k) d\mu_{\omega_k} \right) k^n + \mathcal{O}(k^{n-1})$$

for some positive constant c' independent of k, u, ω_k . From the Riemann-Roch theorem, we have

$$N_k = \text{Vol}(X) k^n + c' \left(\int_X S(\omega_k) d\mu_{\omega_k} \right) k^{n-1} + \mathcal{O}(k^{n-2}).$$

Then, we have

$$\begin{aligned} \frac{w_k}{kN_k} &= \frac{1}{\text{Vol}(X)} \int_X \theta_{u,\omega_k} d\mu_{\omega_k} \\ &\quad + \left(\frac{c'}{\text{Vol}(X)^2} \int_X \theta_{u,\omega_k} (S(\omega_k) - \underline{S}) d\mu_{\omega_k} \right) k^{-1} + \mathcal{O}(k^{-2}). \end{aligned}$$

Since $w_k = 0$ for any k due to $\tau_k(t) \in \text{SL}(V_k)$, we get (28).

From (24), (27) and (28), we have

$$\begin{aligned} \mathcal{F}^{\tilde{\sigma}_k}(u) &= - \left(\frac{N_k}{k^n} F_1(u) + \int_X \tilde{\theta}_{u,\omega_k} \tilde{\theta}_{\nu_1,\omega_k} d\mu_{\omega_k} \right) k^{-1} \\ &\quad - \sum_{j=2}^q \left(\frac{N_k}{k^n} F_j(u) + \int_X \tilde{\theta}_{u,\omega_k} (\tilde{\theta}_{\nu_j,\omega_k} + P_j(\tilde{\theta}_{\nu_i,\omega_k}, \theta'_i)) d\mu_{\omega_k} \right) k^{-j} \\ &\quad - \sum_{j=2}^q \left(\int_X \tilde{\theta}_{u,\omega_k} (\Delta_{\omega_k}(\tilde{\theta}_{\nu_{j-1},\omega_k} + P_{j-1}(\tilde{\theta}_{\nu_i,\omega_k}, \theta'_i))) d\mu_{\omega_k} \right) k^{-j} \\ (29) \quad &\quad - \sum_{j=q+1}^{\infty} \frac{N_k}{k^n} F_j(u) k^{-j} + \mathcal{O}(k^{-q-1}). \end{aligned}$$

Recall that for holomorphy potentials $\tilde{\theta}_1, \tilde{\theta}_2$ under the normalization

$$\int_X \tilde{\theta}_i d\mu = 0,$$

the bilinear form (7) is non-degenerate (see [13]). Then, we can construct ν_j inductively in j so that for $j = 1$

$$\frac{N_k}{k^n} F_1 + \int_X \tilde{\theta}_{u,\omega_k} \tilde{\theta}_{\nu_1,\omega_k} d\mu_{\omega_k} = 0$$

and for $2 \leq j \leq q-1$

$$\begin{aligned} \int_X \tilde{\theta}_{u,\omega_k} \tilde{\theta}_{\nu_j,\omega_k} d\mu_{\omega_k} &= - \frac{N_k}{k^n} F_j \\ &\quad - \int_X \tilde{\theta}_{u,\omega_k} P_j(\tilde{\theta}_{\nu_i,\omega_k}, \theta'_i) d\mu_{\omega_k} \\ &\quad - \int_X \tilde{\theta}_{u,\omega_k} \Delta_{\omega_k}(\tilde{\theta}_{\nu_{j-1},\omega_k} + P_{j-1}(\tilde{\theta}_{\nu_i,\omega_k}, \theta'_i)) d\mu_{\omega_k} \end{aligned}$$

In particular, by definition of extremal vector fields, ν_1 is equal to the extremal vector field. We remark that the right hand side in the above equality are independent of the choice of ω due to [14] and the independency of the k^{-j} ($j \geq 1$) coefficients in (27). Finally, we prove the inequality (23). The construction of ν_j implies

$$(30) \quad |\mathcal{F}^{\tilde{\sigma}_k}(u)| = \left| \sum_{j=q+1}^{\infty} \left(\frac{N_k}{k^n} F_j(u) + \int_X \tilde{\theta}_{u,\omega_k} (\tilde{\theta}_j + \Delta_k \tilde{\theta}_{j-1}) d\mu_{\omega_k} \right) k^{-j} \right|.$$

As we have seen, the coefficients of k^{-j} in the right hand side of (30) are independent of ω , i.e., Lie algebra homomorphisms on the finite dimensional $\text{Lie}(T_k^c)$. Then there

exists a constant C such that

$$\left| \frac{N_k}{k^n} F_j(u) + \int_X \tilde{\theta}_{u, \omega_k} (\tilde{\Theta}_j + \Delta_k \tilde{\Theta}_{j-1}) d\mu_{\omega_k} \right| \leq C \max_i |b_i|$$

for large enough k . Hence we prove (23). The proof is completed. \square

Next we perturb an approximate weight in Lemma 4.7 to the one we desired. Let $v(k)$ be the vector field in Lemma 4.7 for $q = 3n + 3 \geq 3 \log_k N_k + 2$. Then, we connect from σ_k to $\tilde{\sigma}_k$ by a path

$$\zeta_k(t) = \exp(\text{diag}(\frac{1}{2} b_i t))$$

for some basis \mathbf{s} for $\text{Hilb}_k(\omega)$. As seen in the proof of Proposition 4.2 (1),

$$\begin{aligned} \mathcal{F}^{\sigma_k}(v(k)) &= \int_0^1 dt \int_X \frac{\sum_{\alpha} |b_{\alpha}|^2 e^{-tb_{\alpha}} |s_{\alpha}|^2}{\sum_{\beta} |s_{\beta}|^2} d\mu_{\omega_k} \\ (31) \quad &\geq |b_{\alpha_0}|^2 \int_X \frac{|s_{\alpha_0}|^2}{\sum_{\beta} |s_{\beta}|^2} d\mu_{\omega_k} \end{aligned}$$

$$(32) \quad \geq \frac{C}{N_k} |b_{\alpha_0}|^2$$

$$(33) \quad \geq \frac{C}{(N_k)^3} \max |b_{\alpha}|^2$$

for sufficiently large k . In (31), the subscript α_0 is determined so that $b_{\alpha_0} = \inf b_{\alpha} < 0$. The inequality (32) follows from

$$\begin{aligned} \int_X \frac{|s_{\alpha_0}|^2}{\sum_{\beta} |s_{\beta}|^2} d\mu_{\omega_k} &\geq C \int_X \frac{|s_{\alpha_0}|_{h^k}^2}{\sum_{\beta} |s_{\beta}|_{h^k}^2} d\mu_{\omega} \\ &\geq C \frac{1}{\max_X \rho_k(\omega)} \\ &\geq C \frac{1}{\max_X (k^n + A_1(\omega) k^{n-1} + \dots)} \\ &\geq \frac{C}{N_k}. \end{aligned}$$

Recall that $\rho_k(\omega)$ denotes by the k -th Bergman function of ω . In the first line, we use $\omega(k) \rightarrow \omega$ as $k \rightarrow \infty$. In the third line, we use the fact that $\max_X |A_i(\omega)|$ is independent of ω . The inequality (33) follows from

$$|b_{\alpha_0}| \geq \frac{1}{N_k - 1} \max |b_{\alpha}|,$$

because $\sum_{\alpha} b_{\alpha} = 0$.

From (23) and (33), we have

$$(34) \quad \max_{\alpha} |b_{\alpha}| < C k^{-2}.$$

This implies that

$$k(v(k) - v_k) \rightarrow 0$$

as $k \rightarrow \infty$. The proof of Proposition 4.6 is completed. \square

5. PROOF OF THEOREM A

Start with a polarized (X, L) with an extremal metric $\omega_{ex} \in 2\pi c_1(L)$. Here σ_k denotes the optimal weight as defined in Definition 4.3. The key proposition that we want to use is the following (see [7] and [36]):

Proposition 5.1. *Let (Z, ω_Z) be a (finite dimensional) Kähler manifold with a Hamiltonian G -action, for a compact Lie group G . Denote by μ the moment map for this action. For each $x \in Z$, let G_x be the stabilizer of x in G , and let Λ_x^{-1} denote the operator norm of*

$$\sigma_x^* \circ \sigma_x : \text{Lie}(G_x)^\perp \rightarrow \text{Lie}(G_x)^\perp$$

where $\sigma_x : \text{Lie}(G_x)^\perp \rightarrow T_x Z$ is the infinitesimal action of G at x and the orthogonal complement is computed with respect to an invariant scalar product on $\text{Lie}(G)$. Let $x_0 \in Z$ with $\mu(x_0) \in \text{Lie}(G_{x_0})^\perp$. Assume that for some real numbers λ, δ such that

- $\Lambda_x \leq \lambda$ for all $x = e^{i\xi} x_0$ with $\|\xi\| < \delta$, and
- $\lambda \|\mu(x_0)\| < \delta$.

Then there exists $y = e^{i\eta} x_0$ such that $\mu(y) = 0$, with $\|\eta\| \leq \lambda \|\mu(x_0)\|$.

We want to use Proposition 5.1 with the moment map setting of Section 3 to build σ_k -balanced metrics. This will rely on two steps. In Section 5.2 we will estimate the norm of Λ_x in our setting. Then in Section 5.7 we will construct an “almost σ_k -balanced metric” for k large enough.

5.2. Control of the derivative of μ^σ . The aim of this section is to prove:

Proposition 5.3. *For any $R > 1$, there are positive constants C and $\varepsilon < \frac{1}{10}$ such that, for any k , if the basis $\mathbf{s} \in \mathcal{B}^T(V_k)$ has R -bounded geometry, and if $\|\mu^{\sigma_k}(\mathbf{s})\|_{\text{op}} < \varepsilon$, then*

$$(35) \quad \Lambda_{\mathbf{s}} \leq Ck^2.$$

Here $\Lambda_{\mathbf{s}}$ be as in Proposition 5.1 and $\|A\|_{\text{op}}$ denotes the operator norm of a matrix A

$$\|A\|_{\text{op}} := \max \frac{|A(\xi)|}{|\xi|}.$$

This is a generalization of Theorem 21 in [7] and Theorem 2 in [31]. The estimate (35) in Proposition 5.3 is equivalent to

$$\forall \xi \in \text{Lie}(G_{\mathbf{s}})^\perp, \quad g_{Z_k}(\xi, \xi) \geq Ck^{-2} \|\xi\|^2$$

where $G_{\mathbf{s}}$ is the stabilizer of \mathbf{s} in G_k . First we need a nice formulation for g_{Z_k} on $\text{Lie}(G_{\mathbf{s}})^\perp$, which corresponds to (5.5) in [31]. Note that for each $\mathbf{s} \in Z^T(V_k)$,

$$G_{\mathbf{s}}^c = \text{Aut}_k^T$$

and

$$\text{Aut}_k^T \simeq T^c$$

as T is a maximal torus. Thus for any $\mathbf{s} \in Z^T(V_k)$,

$$\text{Lie}(G_{\mathbf{s}})^\perp \simeq \text{Lie}(T)^\perp.$$

We recall that for any $\xi \in \text{Lie}(G_k)$, $\hat{\xi}$ is the induced vector field on \mathbb{CP}^{N_k} . We will denote $\hat{\xi}^\perp$ the orthogonal projection onto the orthogonal of TX in $\Phi_{\mathbf{s}}^* TX$ for the (induced) Fubini-Study metric.

Lemma 5.4. *Let $\xi \in \text{Lie}(G_k)$. Then*

$$g_{\mathcal{B}_k}(\mathbf{s})(\xi, \xi) = \frac{1}{V} \int_X g_{FS}(\widehat{\sigma^* \xi}^\perp, \hat{\xi}^\perp) d\mu_{\phi_s}$$

where $\sigma^* \xi$ is the matrix with entries $\chi_j(\sigma) \xi_{jk} \overline{\chi_k(\sigma)}$ and where for any $A \in \text{Lie}(G_k)$, \hat{A}^\perp denotes the orthogonal projection with respect to the Fubini-Study metric onto the orthogonal of $T(\Phi_s \circ \iota(X))$ in $T\mathbb{CP}^{N_k}$.

Proof. We abbreviate the subscript k for σ_k . Let $\mathbf{s}(t) = \mathbf{s} \cdot \exp(t\xi)$ and $\mathbf{s}(t)^c = \mathbf{s} \cdot \exp(tJ_{\mathcal{B}_k}\xi)$. As $\xi \in \text{Lie}(G_k)$, $e^{t\xi} \in G_k$ and $\omega_\phi(\mathbf{s}(t)) = \omega_\phi(\mathbf{s}(0))$. Then from (14),

$$g_{\mathcal{B}_k}(\xi, \xi) = \left. \frac{d^2}{dt^2} \right|_{t=0} (I^\sigma \circ \phi)(\mathbf{s}(t)^c).$$

Now, using (16) and (17),

$$g_{\mathcal{B}_k}(\xi, \xi) = \left. \frac{d}{dt} \right|_{t=0} - \int_X \sum_{k,j} (\sigma \cdot s_k(t)^c, \sigma \cdot s_j(t)^c)_{h_t^c} \overline{i\xi_{jk}} d\mu_{\phi_{\mathbf{s}(t)^c}}.$$

Note that $\sigma \in \text{Aut}^T = \rho_k(T^c)$, so that σ acts via the characters $\chi \in w_k(T)$. Let us denote by χ_j the character acting on s_j . We obtain

$$\begin{aligned} g_{\mathcal{B}_k}(\xi, \xi) &= \left. \frac{d}{dt} \right|_{t=0} i \int_X \sum_{k,j} (s_k(t)^c, s_j(t)^c)_{h_t^c} \overline{\chi_j(\sigma) \xi_{jk} \chi_k(\sigma)} d\mu_{\phi_{\mathbf{s}(t)^c}} \\ &= \left. \frac{d}{dt} \right|_{t=0} i \int_X \sum_{k,j} (s_k(t)^c, s_j(t)^c)_{h_t^c} \overline{(\sigma^* \xi)_{jk}} d\mu_{\phi_{\mathbf{s}(t)^c}} \\ &= \left. \frac{d}{dt} \right|_{t=0} \langle \mu(\mathbf{s}(t)^c), \sigma^* \xi \rangle \end{aligned}$$

where $\mu(\mathbf{s})$ is the usual moment map

$$\begin{aligned} \mu : \mathcal{B}^T(V_k) &\rightarrow \text{Lie}(G_k) \\ \mathbf{s} &\mapsto i\text{Hilb}(\phi(\mathbf{s}))_0 \end{aligned}$$

Note that $\sigma^* \xi$ is anti-hermitian. Then

$$\begin{aligned} g_{\mathcal{B}_k}(\xi, \xi) &= \left. \frac{d}{dt} \right|_{t=0} \langle \mu(\mathbf{s}(t)^c), (\sigma^* \xi)_0 \rangle \\ &= \frac{1}{V} \int_X g_{FS}(\widehat{(\sigma^* \xi)_0}^\perp, \hat{\xi}^\perp) d\mu_{\phi_s} \\ &= \frac{1}{V} \int_X g_{FS}(\widehat{\sigma^* \xi}^\perp, \hat{\xi}^\perp) d\mu_{\phi_s}. \end{aligned}$$

The two last equalities follow from Lemma 8.4.6 in [16] where the same computations are done with no twisting by σ . Note that the trace part of $\sigma^* \xi$ does not affect the computations as it correspond to a trivial vector field on \mathbb{CP}^{N_k} . \square

To obtain the estimate (35) of Proposition 5.3, we use a comparison between μ^{σ_k} and μ .

Lemma 5.5. *Let σ_k be the optimal weight, and denote by $[\sigma_k]$ the matrix representing σ_k in any basis $\mathbf{s} \in \mathcal{B}^T(V_k)$. There exists a constant $c > 0$ such that for sufficiently large k ,*

$$c^{-1} < \inf_{ij} |[\sigma_k]_{ij}| \leq \sup_{ij} |[\sigma_k]_{ij}| < c$$

Proof. Let $\theta_{v_k, \mathbf{s}}$ be the holomorphy potential function of the vector field v_k satisfying $\sigma_k = \exp(v_k)$ as in (5). Recall that $\theta_{v_k, \mathbf{s}}$ defines the lifted action of v_k on L . Then, we find that there exists a constant $c > 0$ such that

$$(36) \quad \exp(c \min \theta_{v_k, \mathbf{s}}) < \inf_{ij} |[\sigma_k]_{ij}| \leq \sup_{ij} |[\sigma_k]_{ij}| < \exp(c \max \theta_{v_k, \mathbf{s}})$$

for sufficiently large k . From the theory of moment map, both of $\max \theta_{v_k, \omega_{\mathbf{s}}}$ and $\min \theta_{v_k, \omega_{\mathbf{s}}}$ are independent of the choice of \mathbf{s} . In fact, they are determined by the image of the moment map $\mu : X \rightarrow \mathbb{R} = \text{Lie}(S^1)$ with respect to the S^1 -action on $(X, \omega_{\mathbf{s}})$ induced by v_k . On the other hand, from the normalization (6) and Proposition 4.6, we find that

$$\theta_{v_k, \mathbf{s}_k} \rightarrow \theta_{v_{ex}, \omega}$$

as $k \rightarrow \infty$ for a given ω , where $\theta_{v_{ex}, \omega}$ is the potential function satisfying the normalization as (6) and \mathbf{s}_k is an orthonormal basis with respect to $\text{Hilb}_k(\omega)$. Since the maximum and minimum of $\theta_{v_{ex}, \omega}$ are also independent of ω , for sufficiently large k , there exists $C > 0$ such that

$$(37) \quad C^{-1} < \exp(\min \theta_{v_k, \mathbf{s}}) \leq \exp(\max \theta_{v_k, \mathbf{s}}) < C$$

for any \mathbf{s} and k . The equalities (36) and (37) complete the proof. \square

Corollary 5.6. *There exists a constant $c > 0$ independent on k such that for any $\mathbf{s} \in \mathcal{B}^T(V_k)$,*

$$c^{-1} \|\mu(\mathbf{s})\|_{op} < \|\mu^{\sigma_k}(\mathbf{s})\|_{op} < c \|\mu(\mathbf{s})\|_{op}$$

Proof. Note that we compute the operator norm on $\text{Lie}(G)$, so that for any $\mathbf{s} \in \mathcal{B}^T(V_k)$,

$$\|\mu^{\sigma_k}(\mathbf{s})\|_{op} = \|i \text{Hilb}(\sigma^{-1*} \phi(\mathbf{s}))|_{\text{Lie}(G)}\|_{op}$$

and

$$\|\mu(\mathbf{s})\|_{op} = \|i \text{Hilb}(\phi(\mathbf{s}))|_{\text{Lie}(G)}\|_{op}.$$

Then the result follows directly from Lemma 5.5. \square

Proof of Proposition 5.3. We want to prove the following :

$$\forall \xi \in \text{Lie}(T)^\perp, \quad g_{Z_k}(\xi, \xi) \geq Ck^{-2} \|\xi\|^2.$$

Let $\xi \in \text{Lie}(T)^\perp$. Note that for any $t_\xi \in \text{Lie}(T)$,

$$(38) \quad g_{Z_k}(\xi + t_\xi, \xi + t_\xi) = g_{Z_k}(\xi, \xi).$$

In particular, we can chose any lift of ξ in $\text{Lie}(G)$ modulo $\text{Lie}(T)$ to obtain estimates for $g_{Z_k}(\xi, \xi)$. Then by Lemma 5.4 and Lemma 5.5, there is a constant $c > 0$ such that

$$(39) \quad g_{Z_k}(\mathbf{s})(\xi, \xi) \geq c g_{\mathcal{D}}(\mathbf{s})(\xi, \xi)$$

where $g_{\mathcal{D}}$ is the (non-twisted) Kähler metric on $Z^T(V_k)$ defined by Donaldson and used by Phong-Sturm in [31, Theorem 2]. Then, by the same argument as in [31], and using Corollary 5.6, we can estimate $g_{\mathcal{D}}(\mathbf{s})(\xi, \xi)$. The context to use the arguments of Phong-Sturm [31] is identical except that we are in the presence of

holomorphic vector fields. As we work modulo T , the remaining vector fields are precisely in $\text{Lie}(\text{Aut}^T) \simeq \text{Lie}(T^c)$. The only point that is used in [31] and fails because of the existence of holomorphic vector fields is a $\bar{\partial}$ estimate for vector fields. More precisely, the following fails in general:

$$(40) \quad \|w\|_{L^2_1(\omega_0)} \leq C \|\bar{\partial}w\|_{L^2(\omega_0)}$$

for some positive constant C , with $w \in H$, where H is the L^2_1 -completion of the space of complex T -invariant Hamiltonian vector fields. This is used to show equation (5.9) in [31]. What is true in our setting is that (40) holds for all

$$w \in \ker(\bar{\partial}|_H)^{\perp_{L^2}} \simeq \text{Lie}(T)^{\perp_{L^2}},$$

where the orthogonal is computed with respect to the L^2 inner product given by integration over X , with the metric ω_s on the base and the metric ω_{FS} on the fiber. We claim that there exists a unique $t_\xi \in \text{Lie}(T)$ such that

$$\xi + t_\xi \in \text{Lie}(T)^{\perp_{L^2}}.$$

Indeed, if $\{t_1, \dots, t_r\}$ is a basis for $\text{Lie}(T)$, t_ξ is the unique solution to

$$\forall i \in \{1, \dots, r\}, \quad \int_X g_{FS}(\hat{t}_\xi, \hat{t}_i) d\mu_{\phi(s)} = - \int_X g_{FS}(\hat{\xi}, \hat{t}_i) d\mu_{\phi(s)}$$

which exists as

$$(t_1, t_2) \mapsto \int_X g_{FS}(\hat{t}_1, \hat{t}_2) d\mu_{\phi(s)}$$

defines a positive definite symmetric bilinear form on $\text{Lie}(T)$. Then, we can apply the estimates from [31] to $\xi + t_\xi$:

$$g_{\mathcal{D}}(s)(\xi + t_\xi, \xi + t_\xi) \geq C' \|\xi + t_\xi\|^2 k^{-2}.$$

But $\xi \in \text{Lie}(T)^\perp$, so $\|\xi + t_\xi\|^2 = \|\xi\|^2 + \|t_\xi\|^2$ and

$$g_{\mathcal{D}}(s)(\xi + t_\xi, \xi + t_\xi) \geq C' \|\xi\|^2 k^{-2}.$$

Using (38) and (39), we conclude that there is $c' > 0$ such that

$$g_{Z_k}(s)(\xi, \xi) \geq c' k^{-2} \|\xi\|^2.$$

□

5.7. Construction of almost σ -balanced metrics. In this section we prove the following theorem to obtain the approximated σ -balanced metrics.

Theorem 5.8. *Let ω_{ex} be a T -invariant extremal metric in the class $2\pi c_1(L)$, where $T \subset \text{Aut}_0(X)$ is a maximal compact torus. Let σ_k be the optimal weights associated with this torus. Then there are T -invariant functions $\eta_j \in C^\infty(X, \mathbb{R})^T$ such that for each $q > 0$ the metrics*

$$\omega_q(k) = \omega_\infty + i\partial\bar{\partial}\left(\sum_{j=1}^q \eta_j k^{-j}\right)$$

satisfy the following:

$$(41) \quad k^{-n} \rho_k(\omega_q(k)) = \exp(\psi_k(\omega_q(k))) + \mathcal{O}(k^{-q-2})$$

First, we show the following expansion of $\exp(\psi_k(\omega))$ for a given ω .

Proposition 5.9. *Let ω be a T -invariant metric. There exist T -invariant functions $B_j(\omega)$ such that for each $q > 0$*

$$\exp(\psi_{\sigma_k, \omega}) = \sum_{j=0}^q k^{-j} B_j(\omega) + e_q(\omega, k)$$

satisfying that for any $l \in \mathbb{N}$, there is a constant $C_{l,q}$ such that

$$\|e_q(\omega, k)\|_{C^l} \leq C_{l,q} k^{-q-1}.$$

Proof. From the proof of Proposition 4.6, we find that for each $q > 0$

$$v_k = \sum_j \nu_j k^{-j} + \mathcal{O}(k^{-q-1})$$

where ν_j defined in Lemma 4.7. In fact, we can get the estimate (34) for any power in k by increasing q in Lemma 4.7. Then for any T -invariant metric ω_ϕ , we deduce a uniform expansion in $C^l(X, \mathbb{R})$ -topology in the space \mathcal{H} for

$$\sigma_k^* \omega_\phi - \omega_\phi = i\partial\bar{\partial}\psi_{\sigma_k, \phi} = i\partial\bar{\partial} \sum_j^q \theta_j k^{-j} + \mathcal{O}(k^{-q-1})$$

as in (26). From this we deduce the expansion for $\exp(\psi_{\sigma_k, \phi})$. \square

We will need the following Lemmas:

Lemma 5.10. *Let ω be a T -invariant metric. Then*

$$B_0(\omega) = 1, \quad B_1(\omega) = \frac{1}{2}(\theta_{ex, \omega} + \underline{S})$$

where $\theta_{ex, \omega}$ is the holomorphy potential of the extremal vector field with respect to ω . Moreover, if ω is extremal

$$4D_\omega B_1(\phi) = \nabla\phi \cdot \nabla S(\omega)$$

Proof. The first statement follows from that ν_1 is equal to v_{ex} as Lemma 4.7. The second statement follows from the computation of the differential of B_1 and is a standard computation, see e.g. [16, Lemma 5.2.9]. \square

Lemma 5.11. *Let ω be any T -invariant metric. Then for any $v \in \text{Lie}(T^c)$,*

$$(42) \quad \int_X \tilde{\theta}_{v, \omega} (1 + k^{-1} \Delta_\omega) e^{\psi_{\sigma_k, \omega}} d\mu_\omega = \int_X \tilde{\theta}_{v, \omega} (1 + k^{-1} \Delta_\omega) k^{-n} \rho_k(\omega) d\mu_\omega$$

where $\tilde{\theta}_{v, \omega}$ is the mean value zero holomorphy potential of v with respect to ω .

Proof. Note that through this proof, the Laplacian considered is the complex Laplacian while in [16] this is the d -Laplacian. From the choice of the weights σ_k , we have

$$\mathcal{F}^{\sigma_k}(v) = 0$$

thus

$$\int_X \theta_{v, \omega} (1 + k^{-1} \Delta_\omega) e^{\psi_{\sigma_k, \omega}} d\mu_\omega = 0$$

for any $v \in \text{Lie}(T^c)$ and any T -invariant ω that is a pullback of the Fubini-Study metric. We recall (from the proof of Proposition 4.2) that there is a constant c_k depending on k such that $\theta_{v, \omega} = c_k + \tilde{\theta}_{v, \omega}$, where $\tilde{\theta}_{v, \omega}$ has mean value zero. Then,

$$\int_X \tilde{\theta}_{v, \omega} (1 + k^{-1} \Delta_\omega) e^{\psi_{\sigma_k, \omega}} d\mu_\omega = -c_k \frac{N_k}{k^n}.$$

Note that the above equation makes sense for any T -invariant metric (even non pulled-back metrics). We now consider the action induced by v on $H^0(X, L^k)$ (see [16, Proposition 8.6.1 page 200]). We obtain

$$k^{-(n+1)}w_k = \int_X (1 + k^{-1}\Delta_\omega)\theta_{v,\omega}k^{-n}\rho_k(\omega)d\mu_\omega.$$

As we lift the v action into $SL(H^0(X, L^k))$, the weight vanishes and we have

$$(43) \quad \int_X (1 + k^{-1}\Delta_\omega)\tilde{\theta}_{v,\omega}k^{-n}\rho_k(\omega)d\mu_\omega = -c_k \frac{N_k}{k^n}$$

for any T -invariant metric. The result follows. \square

Proof of Theorem 5.8. In the following, we only consider T -invariant functions. We will omit the subscript T , but we shall keep in mind that all the functions considered are supposed to be T -invariant. In particular, if \mathbb{L}_g is the Lichnerowicz operator, we restrict to $\ker(\mathbb{L}_g)^T$, that is to T -invariant Killing potentials. As T is maximal, these potentials are exactly the Killing potentials of the elements of $\text{Lie}(T)$. The proof is by induction on q . Write down the expansions

$$k^{-n}\rho_k(\omega_{ex} + i\partial\bar{\partial}\eta) = \sum_{j=0}^{\infty} A_j(\omega_{ex} + \eta)k^{-j}$$

and

$$\exp(\psi_k(\omega_{ex} + i\partial\bar{\partial}\eta)) = \sum_{j=0}^{\infty} B_j(\omega_{ex} + \eta)k^{-j}$$

where we set

$$\eta := \sum_{l=1}^q \eta_l k^{-l}.$$

We use the Taylor expansions of the coefficients A_j and B_j to obtain

$$k^{-n}\rho_k(\omega_{ex} + i\partial\bar{\partial}\eta) = \sum_{j=0}^{\infty} A_j(\omega_{ex})k^{-j} + \sum_{j,l} A_{j,l}(\eta)k^{-j-l}$$

and

$$\exp(\psi_k(\omega_{ex} + i\partial\bar{\partial}\eta)) = \sum_{j=0}^{\infty} B_j(\omega_{ex})k^{-j} + \sum_{j,l} B_{j,l}(\eta)k^{-j-l}$$

where the $A_{j,l}(\eta)$ and $B_{j,l}(\eta)$ are polynomial expressions in the η_l and their derivatives, depending on ω_{ex} . Assume that the T -invariant functions $(\eta_j)_{j \leq q-1}$ are chosen so that the above expansions agree till order q . We try to choose η_q so that the expansions agree till order $q+1$. The coefficients of order $k^{-(q+1)}$ in the two expansions are

$$A_{q+1}(\omega_{ex}) + \sum_{I_{q+1}} A_{j,l}(\eta_1, \dots, \eta_{q-1}) + \frac{1}{2}DS_{\omega_{ex}}(\eta_q)$$

and

$$B_{q+1}(\omega_{ex}) + \sum_{I_{q+1}} B_{j,l}(\eta_1, \dots, \eta_{q-1}) + \frac{1}{4}\nabla\eta_q \cdot \nabla S(\omega_{ex})$$

where we used the fact that ω_{ex} is extremal together with Lemma 5.10. Here the sets of indices I_{q+1} are defined by the above expressions. Then the terms of order $q+1$ will agree if and only if we have

$$(44) \quad \frac{1}{2}\mathbb{L}_{\omega_{ex}}(\eta_q) = A_{q+1}(\omega_{ex}) - B_{q+1}(\omega_{ex}) + \sum_{I_{q+1}} (A_{j,l} - B_{j,l})(\eta_1, \dots, \eta_{q-1})$$

where \mathbb{L}_g is the Lichnerowicz operator of any metric g . The equation (44) has a solution if and only if

$$(45) \quad A_{q+1}(\omega_{ex}) - B_{q+1}(\omega_{ex}) + \sum_{I_{q+1}} (A_{j,l} - B_{j,l})(\eta_1, \dots, \eta_{q-1}) \in \ker(\mathbb{L}_{\omega_{ex}})^\perp.$$

We cannot say much about (45), but it only depends on $\eta_1, \dots, \eta_{q-1}$ so we will add in the recursive process the assumption that at each step, (45) is satisfied. Then equation (44) can be solved recursively. Note that the initialization of the process requires

$$(46) \quad A_2 - B_2(\omega_{ex}) \in \ker(\mathbb{L}_{\omega_{ex}})^\perp.$$

To simplify notations, set

$$R_{q+2}(\eta_1, \dots, \eta_q) = A_{q+2}(\omega_{ex}) - B_{q+2}(\omega_{ex}) + \sum_{I_{q+2}} (A_{j,l} - B_{j,l})(\eta_1, \dots, \eta_q).$$

It remains to show (46) and that, when solving (44), we can choose η_q so that the following is true:

$$(47) \quad R_{q+2}(\eta_1, \dots, \eta_q) \in \ker(\mathbb{L}_{\omega_{ex}})^\perp.$$

We now apply Lemma 5.11 to

$$\omega_\eta := \omega_{ex} + i\partial\bar{\partial}\eta = \omega_{ex} + i\partial\bar{\partial}\sum_{l=1}^q \eta_l k^{-l}.$$

Equation (42) can be written

$$(48) \quad \int_X \tilde{\theta}_{v,\omega_\eta} (1 + k^{-1}\Delta_{\omega_\eta})(k^{-n}\rho_k(\omega_\eta) - e^{\psi_{\sigma_k,\omega_\eta}}) d\mu_{\omega_\eta} = 0.$$

Then, by the induction hypothesis (choice of η_1, \dots, η_q), we have the following expansion:

$$(49) \quad k^{-n}\rho_k(\omega_\eta) - e^{\psi_{\sigma_k,\omega_\eta}} = R_{q+2}(\eta_1, \dots, \eta_q)k^{-(q+2)} + \mathcal{O}(k^{-(q+3)})$$

We also have:

$$\omega_\eta = \omega_{ex} + \mathcal{O}(k^{-1}).$$

Thus we deduce with (49) in equation (48), that the term of order $k^{-(q+2)}$ in the expansion vanishes, that is

$$\int_X \tilde{\theta}_{v,\omega_{ex}} R_{q+2}(\eta_1, \dots, \eta_q) d\mu_{\omega_{ex}} = 0.$$

Note also that the above argument with $\eta = 0$ gives (46). The proof is complete. \square

5.12. Completion of Proof of Theorem A. Once we have Proposition 5.3 and Theorem 5.8, the proof of Theorem A is almost identical to [7]. We give the outline of the proof. Fix an arbitrary $R > 1$. Fix an integer q determined later. For the Kähler form $\omega_q(k)$ in Theorem 5.8, we have

$$k^{-n} \rho_k(\omega_q(k)) = \exp(\psi_k(\omega_q(k)))(1 + \epsilon_k)$$

where $\epsilon_k = \mathcal{O}(k^{-q-2})$. Let

$$\omega'(k) := \omega_q(k) + i\partial\bar{\partial} \log(\exp(\psi_k(\omega_q(k)))(1 + \epsilon_k)) = \omega_{\mathbf{s}_0}$$

where \mathbf{s}_0 is an orthonormal basis with respect to $\text{Hilb}_k(\omega_q(k))$. From Proposition 27 in [7], for large k , we find that there exists some (small) constant $c > 0$ depending only on R such that if $a \in \text{Lie}(G_k)$ satisfies $\|a\|_{\text{op}} < c$, then

- (1) $\mathbf{s}_0 \cdot e^{ia}$ is R -bounded, and
- (2) there exists C_1 such that

$$\|[\mu^{\sigma_k}(\mathbf{s}_0 \cdot e^{ia})]\|_{\text{op}} \leq C_1(\|a\|_{\text{op}} + \|\epsilon_k\|_{C^2, \omega_{ex}}).$$

In particular, we have

$$\|[\mu^{\sigma_k}(\mathbf{s}_0)]\|_{\text{op}} \leq C_2 k^{-q-2}.$$

Proposition 5.3 implies that if a satisfies

$$C_1(\|a\|_{\text{op}} + \|\epsilon_k\|_{C^2, \omega_{ex}}) < \varepsilon$$

where ε is defined in Proposition 5.3, then

$$\Lambda_{\mathbf{s}_0 \cdot e^{ia}} \leq C_3 k^2$$

for some C_3 . Now, we will apply Proposition 5.1 by putting $Z := Z^T(V_k)$ with ω_Z defined in Proposition 3.4, $G := G_k$ defined in (10) and $\mu := \mu^{\sigma_k}$. Let δ in Proposition 5.1 be

$$\min(c, \frac{\varepsilon}{2C_1})$$

where c, C_1 are as above. If

$$\mu^{\sigma_k}(\mathbf{s}_0) = g_1 + g_2, \quad g_1 \in \text{Lie}(G_{\mathbf{s}_0}), \quad g_2 \in \text{Lie}(G_{\mathbf{s}_0})^\perp,$$

then we replace \mathbf{s}_0 by $\mathbf{s}_0 \cdot e^{-ig_1}$. Then, we can assume that

$$\mu^{\sigma_k}(\mathbf{s}_0) \in \text{Lie}((G_k)_{\mathbf{s}_0})^\perp.$$

From Proposition 4.2 and $\text{Lie}(G_{\mathbf{s}_0}) = \text{Lie}(T_k)$, we can assume that the inequality

$$\|[\mu^{\sigma_k}(\mathbf{s}_0)]\|_{\text{op}} \leq C_2 k^{-q-2}$$

still holds. Putting $\lambda := C_3 k^2$,

$$\lambda \|[\mu^{\sigma_k}(\mathbf{s}_0)]\| \leq \sqrt{N_k} \lambda \|[\mu^{\sigma_k}(\mathbf{s}_0)]\|_{\text{op}} < C_2 C_3 k^{n/2-q}.$$

Taking q so that $n/2 - q < 0$, for large k , we have

$$\lambda \|[\mu^{\sigma_k}(\mathbf{s}_0)]\| < \delta.$$

Proposition 5.1 implies that there exists $a \in \text{Lie}((G_k)_{\mathbf{s}_0})^\perp$ such that

$$\mu^{\sigma_k}(\mathbf{s}_0 \cdot e^{ia}) = 0, \quad \|a\| \leq C_2 C_3 k^{n/2-q},$$

i.e., $\mathbf{s}_0 \cdot e^{ia}$ is σ_k -balanced point we desired. By construction, considering the behavior of C^r -norm by scaling $\omega \mapsto k\omega$,

$$\|\omega_{\phi_{\mathbf{s}_0 \cdot e^{ia}}} - \omega_{ex}\|_{C^r} = \mathcal{O}(k^{n/2-q+r}).$$

For any $r \geq 0$, by replacing q so that $n/2 - q + r < 0$, we proved that σ_k -balanced metrics $\omega_{\phi_{\mathbf{s}_0, e^{ia}}}$ converge to ω_{ex} in C^r -sense. The proof of Theorem A is completed.

5.13. Proofs of Corollaries C and D. We sketch the proofs of Corollaries C and D, that follow from the arguments in [7] and [1] respectively. Let ω be an extremal metric on (X, L) . By Theorem A, ω is a limit of σ_k -balanced metrics.

The proof of Corollary C is as in [7]. A σ_k -balanced metric corresponds to a zero of the moment map μ^{σ_k} . From general theory of moment maps, such a zero is unique, up to the G_k -action, in its G_k^c orbit. The result follows at the limit.

The proof of Corollary D follows the strategy from [1]. Each σ_k -balanced metric is a product of σ_k -balanced metrics on each factor of (X, L^k) . To prove the splitting for σ -balanced metrics, we use the corresponding notion of GIT. The existence of a σ -balanced metric corresponds to the vanishing of a finite dimensional moment map, and to a GIT stability condition. Then we use the general fact that stability for a product implies stability for each factor. Indeed, by Hilbert-Mumford criterion, one has to check stability with respect to one parameter subgroups. But the set of one-parameter subgroups considered for the product contains the one parameter subgroups considered for each factor. We deduce from this that $(X_i, L_i^{\otimes k})$ admits a σ_k -balanced metric for large k , and by unicity, the product of these metrics is our initial σ_k -balanced metric. Then the result follows at the limit.

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